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Total Distance Vertex Irregularity Strength of Hairy Cycle C_m^n Graph

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Abstract

A total graph labeling is an assignment of integers to the union of vertices and edges to certain conditions. The labeling becomes D -distance vertex irregular total k-labeling when each vertex of G has a different weight (which is determined by D-distance neighborhood). The total distance vertex irregularity strength of G denoted by $tdis(G)$ and define as the minimum of the biggest label k over all D-distance vertex irregular total k-labelings of G. In this paper, we investigate about D-distance vertex irregular total k-labelings on *hairy cycle* C_m^n graphs which can be applied to cryptography and computational networks. The unique hairy cycle graph construction makes the weights of each vertex of this graph different and random. Therefore, this weight formula can be applied in stream cipher cryptography as a key generator. To obtained the formula labeling, we carried out labeling experiments repeatedly to find labeling patterns and then formulate it into a labeling function. We also provide the lower bound and determine the value of total distance vertex irregularity strength of *hairy cycle* C_m^n graphs. we prove that for $m = 2,3,4$, $n \geq 5$ an odd positive integer , hairy cycle C_m^n graphs admits an D-distance vertex irregular total k-labelings with total distance vertex irregularity strength, $tdis(C_m^n) = \left\lceil \frac{mn+1}{2} \right\rceil$.

Keywords

Distance Vertex Irregular Total K-Labeling, Hairy Cycle Graphs, The Total Distance Vertex Irregularity Strength

Introduction

Let G be a pair set $\big(V(G), E(G)\big)$, where $V(G)$ is a finite non-empty set of elements called vertex, and $E(G)$ is a finite (possibly empty) set which is an unordered pair u, v of vertex $u, v \in V$ called edges with order $|V(G)| = p$ and size $|V(G)| = q$. Graph labeling was defined by (Wallis, 2000) as a mapping from the set of elements in a graph to a set of numbers, usually positif integers. Based on the domain, labelings are divided into three parts i.e. vertex labeling , edge labeling and total labeling (Parkhurst, 2014). For complete survey on labeling, see (Gallian, 2018).

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One of vertex labelings that was introduced by (Slamin, 2017) is the distance irregular vertex labeling. This labelings, inspire from the concept of distance magic labeling by (Miller, 2003) and defined as a mapping $f: V(G) \rightarrow \{1,2, \dots, k\}$ such that the weight of every vertex $v \in V(G)$ is distinct. The weight v under labeling f is $w_f(v) = \sum_{u \in N(v)} f(u)$, where $N(v)$ is the set of all vertices which is adjacent to v. Hairy cycle graph is a general construction given by G and H graph to produced a corona graph $G \odot H$ (Wojciechowski, 2006). The corona of graphs G_1 and G_2, denoted as $G_1 \odot G_2$, is a graph obtained by taking a duplicate of graphs G_1 and $|V(G_1)|$ duplicate of graph G_2 i.e. G_{2_i} for $i = 1,2, \dots, |V(G_1)|$, then connect the *i-th* vertex of G_1 to each vertex in G_{2_i} (Marzuki, 2018). The idea of distance vertex irregular total k-labeling was first introduced by (Wijayanti, 2020) as a new concept of total labeling based on both vertex irregular total k-labeling and distance vertex irregular labeling. A graph admits distance vertex irregular total k-labeling if there exist a function $f: V(G) \cup E(G) \rightarrow \{1,2, \dots, k\}$ such as for every $u, v \in V(G)$ and $u \neq v$, the weight of u is not equal to v. Formally, (Wijayanti, 2021) defined the distance vertex irregular total k-labeling as follows.

Definition 1. Let $G(V, E)$ be a simple connected graph with $|V(G)| = p$ and $|E(G)| = p$. A distance vertex irregular total k-labeling is a function $f: V(G) \cup E(G) \rightarrow \{1,2, \dots, k\}$ such that the weight of every vertex of G is distinct. The weight of $v \in V(G)$ under the labeling f is defined as

$$w_f(v) = \sum_{u \in N(v)} f(u) + \sum_{uv \in I(v)} f(uv)$$

The total distance vertex irregularity strength of G, denoted by $tdis(G)$ and defined as the smallest value of k for which G has a distance vertex irregular total k-labeling. The lower bound of total distance vertex irregularity strength for a graph G given by (Wijayanti, 2020) in the following lemma.

Lemma 2. Let $G(V, E)$ be a simple connected graph with maximum degree Δ and minimum degree δ. The lower bound of $tdis(G)$ is

$$tdis(G) \geq \left\lceil \frac{|V(G) - 1 + 2\delta|}{2\Delta} \right\rceil$$

As a new concept of irregular labeling, distance vertex irregular total k-labeling has many advantages. The weight of each node which calculated by distance, giving this labeling flexibility to apply. Distance vertex irregular total k-labeling on several types of graphs will produce random vertex weights. This causes the vertex weight function can be applied to cryptography as a keystream generator. The unique of hairy cycle graph construction allows this type of graph to have random point weights. From this description, it is very interesting to investigate distance labeling on hairy cycle graphs. Next, we will use the theorems above to prove some theorems in the main result.

Materials and Methods

In this section, we define the distance vertex irregular total k-labeling and determine the total distance vertex irregularity strength of hairy cycle. To do so, first, we carried out labeling experiments repeatedly to find the labeling patterns of the hairy cycle graph and formulate them into a labeling function. Hairy cycle graph C_m^n is the corona product of C_n and $\overline{K}_m$, denoted as

$C_n \odot \overline{K}_m$. Hence, C_m^n is obtained by taking a duplicate of graphs C_n and $|V(C_n)|$ duplicate of graph $\overline{K}_m$ i.e. K_{m_i} for $i = 1,2,\dots,|V(G_1)|$, then connect the i-th vertex of C_n to each vertex in K_{m_i}. This result in C_m^n has $(m+1)n$ vertices and $(m+1)n$ edges. mn vertices of degree 1 and n vertices of degree $m+2$. The example of hairy cycle graph C_4^n shown in Figure 1.

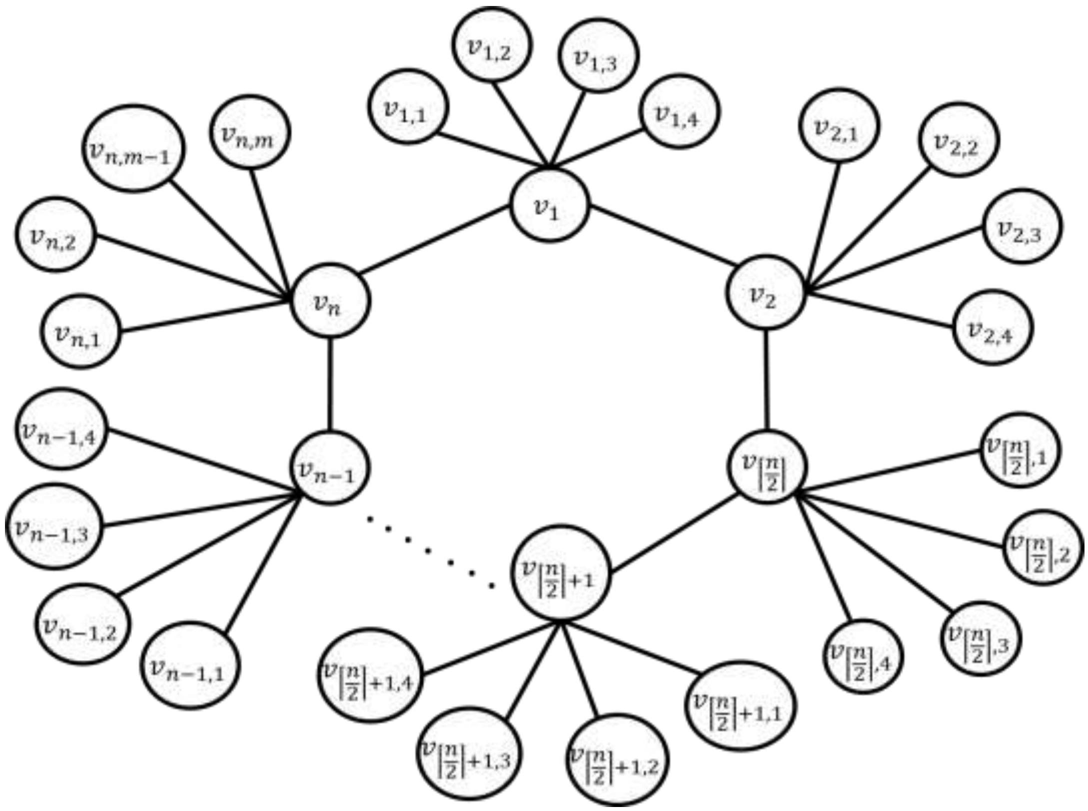

Figure 1. Hairy cycle graph C_m^n with $m = 4$

In Figure 1, the labeling of the vertices of graph G starts from a vertex of degree $m+2$ ($v_i, i = 1,\dots,n.$), then continues with the edge that connects the vertex v_i to v_{ij}. More details, we construct some theorems and then prove them through the labeling function that has been formulated.

The following theorems gives a lower bound of the total distance vertex irregularity strength of C_m^n graph.

Theorem 3. Let C_m^n be a hairy cycle graph with $m = 2,3,4$, $n \geq 5$ an odd positive integer. If C_m^n has the distance vertex irregular total k-labeling then

$$tdis(C_m^n) \geq \left\lceil \frac{mn+1}{2} \right\rceil$$

Proof. Let C_m^n be a hairy cycle graph with with $m = 2,3,4$, $n \geq 5$ and n odd positive integer. Define f as a distance vertex irregular total k-labeling on C_m^n. C_m^n has minimum degree $\delta = 1$, and maximum degree$\Delta = m+2$, with order $|V(C_m^n)| = (m+1)n$ and size $|E(C_m^n)| = (m+1)n$. $V(C_m^n) = \{v_1,\dots,v_n,v_{1,1},\dots,v_{1,m},v_{2,1},\dots,v_{2,m},\dots,v_{n,1},\dots v_{n,m}\}$ and $E(C_m^n) = \{v_1v_2,\dots,v_nv_1,v_1v_{1,1},\dots v_1v_{1,m},v_2v_{2,1},\dots,v_2v_{2,m},v_nv_{n,1},\dots,v_nv_{n,m}\}$. We obtain,

$$tdis(C_m^n) \geq max\left\{\left\lceil\frac{mn+1+n}{2(m+2)}\right\rceil, \left\lceil\frac{mn+1}{2}\right\rceil\right\}$$

For each corresponding value of i, $\left\lceil\frac{mn+1+n}{2(m+2)}\right\rceil \geq \left\lceil\frac{mn+1}{2}\right\rceil$ thus, by definition of distance vertex irregular total k-labeling, the vertices weight are distinct. Consequently, the lower bound of the largest label of hairy cycle graph C_m^n is

$$tdis(C_m^n) \geq \left\lceil\frac{mn+1}{2}\right\rceil$$

Theorem 4. Let C_m^n be a hairy cycle graph with $m = 2$, $n \geq 5$ and n an odd positive integer if C_2^n has a distance vertex irregular total k-labeling then the total distance vertex irregularity strength denoted by $tdis(C_2^n)$ is

$$tdis(C_2^n) = \left\lceil\frac{2n+1}{2}\right\rceil$$

Proof. Let C_2^n be a hairy cycle graph, $V(C_2^n) = \{v_1, \dots, v_n, v_{1,1}, v_{1,2}, \dots, v_{n,1}, v_{n,2}\}$ dan $E(C_2^n) = \{v_1v_2, \dots, v_nv_1, v_1v_{1,1}, v_1v_{1,2}, \dots, v_nv_{n,1}, v_nv_{n,2}\}$ is the set of vertices and edge C_2^n with $|V(C_2^n)| = 3n$ and $|E(C_2^n)| = 3n$. C_2^n has minimum degree $\delta = 1$, and maximum degree $\Delta = 4$. Based on lemma 2, the lower bound of $tdis(C_2^n)$ is

$$tdis(C_2^n) \geq \left\lceil\frac{mn+1}{2}\right\rceil = \left\lceil\frac{2n+1}{2}\right\rceil$$

Define f, a distance vertex irregular total k-labeling on C_2^n as follows :

$$f(v_i) = \begin{cases} 2i-1 & ; i = 1,2,\dots,\left\lceil\frac{n}{2}\right\rceil \\ 4+2(n-i) & ; i = \left\lceil\frac{n}{2}\right\rceil + 1, \dots, n \end{cases}$$

$$f(v_{i,j}) = \frac{n+1}{2} - 2 \qquad ; i = 1,\dots,n \quad j = 1,2$$

$$f(v_iv_{i,j}) = \begin{cases} 2i-3 & ; i = 2,\dots,\left\lceil\frac{n}{2}\right\rceil \quad j = 1 \\ 2i-2 & ; i = 2,\dots,\left\lceil\frac{n}{2}\right\rceil \quad j = 2 \end{cases}$$

$$f(v_iv_{i,j}) = \begin{cases} 2(n-i)+3 & ; i = \left\lceil\frac{n}{2}\right\rceil + 1,\dots,n \quad j = 1 \\ 2(n-i)+2 & ; i = \left\lceil\frac{n}{2}\right\rceil + 1,\dots,n \quad j = 2 \end{cases}$$

$$f(v_iv_{i,j}) = j \qquad ; i = 1 \quad j = 1,2$$

Since $w_f(v_i) = \sum_{v_j \epsilon N(v_i)} f(v_j) + \sum_{v_iv_{i,j} \epsilon I(v_i)} f(v_iv_{i,j})$, for $i = 1,2\dots,n$, $j = 1,2$, $n \geq 5$ n odd positive integer, we obtain,

$$w_f(v_{1,1}) = f(v_1) + f(v_1 v_{1,1}) = 1 + 1 = 2$$
$$w_f(v_{1,2}) = f(v_1) + f(v_1 v_{1,2}) = 1 + 2 = 3$$
$$w_f(v_{2,1}) = f(v_2) + f(v_2 v_{2,1}) = 3 + 1 = 4$$
$$w_f(v_{2,2}) = f(v_2) + f(v_2 v_{2,2}) = 3 + 2 = 5$$
$$\vdots$$
$$\begin{aligned} w_f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1,1}\right) &= f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1} v_{\left\lceil\frac{n}{2}\right\rceil+1,1}\right) \\ &= 4 + 2\left(n - \left\lceil\frac{n}{2}\right\rceil + 1\right) + 2\left(n - \left\lceil\frac{n}{2}\right\rceil + 1\right) + 3 \\ &= 2n + 1 \end{aligned}$$
$$\begin{aligned} w_f(v_1) &= f(v_n) + f(v_n v_1) + f(v_1 v_2) + f(v_2) + f(v_1 v_{1,1}) + f(v_1 v_{1,2}) + f(v_{1,1}) \\ &\quad + f(v_{1,2}) \\ &= 14 \end{aligned}$$
$$\begin{aligned} w_f(v_2) &= f(v_1) + f(v_1 v_2) + f(v_2 v_3) + f(v_3) + f(v_2 v_{2,1}) + f(v_2 v_{2,2}) + f(v_{2,1}) \\ &\quad + f(v_{2,2}) \\ &= 12 \end{aligned}$$
$$\begin{aligned} w(v_n) &= f(v_{n-1}) + f(v_{n-1} v_n) + f(v_1 v_n) + f(v_1) + f(v_n v_{n,1}) + f(v_n v_{n,2}) + f(v_{n,1}) + \\ &\quad + f(v_{n,2}) \\ &= 16 \end{aligned}$$
$$\vdots$$
$$\begin{aligned} w_f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) &= f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1,1}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1} v_{\left\lceil\frac{n}{2}\right\rceil+1,1}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1,2}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1} v_{\left\lceil\frac{n}{2}\right\rceil+1,2}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil}\right) \\ &\quad + f\left(v_{\left\lceil\frac{n}{2}\right\rceil} v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+2}\right) + f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1} v_{\left\lceil\frac{n}{2}\right\rceil+2}\right) \\ &= \frac{n+1}{2} - 2 + 2\left(n - \left(\left\lceil\frac{n}{2}\right\rceil + 1\right)\right) + 3 + \frac{n+1}{2} - 2 + 2\left(n - \left(\left\lceil\frac{n}{2}\right\rceil + 1\right)\right) + 2 \\ &\quad + 2\left(\left\lceil\frac{n}{2}\right\rceil\right) - 1 + \frac{n+1}{2} - 2 + 4 + 2\left(n - \left(\left\lceil\frac{n}{2}\right\rceil + 2\right)\right) + \frac{n+1}{2} - 2 \end{aligned}$$

For n odd, we obtain

$$w_f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) = 6n - 8$$

Thus, for $i = 1, \dots, n$ we obtain the set of vertices weights is $w_f(v_i) = \{2,3, \dots, 2n+1\}$. The vertices weight are distinct and formed an arithmetic progressive, consequently, $k = \left\lceil\frac{2n+1}{2}\right\rceil$.

Theorem 5. Let C_m^n be a hairy cycle graph with $m = 3$, $n \geq 5$ and n an odd positive integer. If C_3^n has a distance vertex irregular total k-labeling then the total distance vertex irregularity strength denoted by $tdis(C_3^n)$ is

$$tdis(C_3^n) = \left\lceil\frac{3n+1}{2}\right\rceil$$

Proof. Let C_3^n be a hairy cycle graph, $V(C_3^n) = \{v_1, \dots, v_n, v_{1,1}, \dots, v_{1,3}, \dots, v_{n,1}, \dots, v_{n,3}\}$ dan $E(C_3^n) = \{v_1 v_2, \dots, v_n v_1, v_1 v_{1,1,}, \dots, v_1 v_{1,3}, \dots, v_n v_{n,1}, \dots, v_n v_{n,3}\}$ is the set of vertices and edge C_3^n

with $|V(C_3^n)| = 4n$ and $|E(C_3^n)| = 4n$. C_2^n has minimum degree $\delta = 1$, and maximum degree $\Delta = 5$. Based on lemma 2, the lower bound of $tdis(C_3^n)$ is

$$tdis(C_3^n) \geq \left\lceil \frac{mn+1}{2} \right\rceil = \left\lceil \frac{3n+1}{2} \right\rceil$$

Define f as a distance vertex irregular total k-labeling on C_3^n as follows :

$$f(v_i) = \begin{cases} 3i-2 & ; i = 1,2,\dots,\left\lceil \frac{n}{2} \right\rceil \\ 5+3(n-i) & ; i = \left\lceil \frac{n}{2} \right\rceil + 1,\dots,n \end{cases}$$

$$f(v_{i,j}) = \frac{n+1}{2} - 3 + \left\lceil \frac{n}{10} \right\rceil \qquad ; i = 1,\dots,n \quad j = 1,2,3$$

$$f(v_i v_{i,j}) = \begin{cases} 3i-5 & ; i = 2,\dots,\left\lceil \frac{n}{2} \right\rceil \\ & \;\; j = 1 \\ 3i-4 & ; i = 2,\dots,\left\lceil \frac{n}{2} \right\rceil \\ & \;\; j = 2 \\ 3i-3 & ; i = 2,\dots,\left\lceil \frac{n}{2} \right\rceil \\ & \;\; j = 3 \end{cases}$$

$$f(v_i v_{i,j}) = \begin{cases} 3(n-i)+5 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1,\dots,n \\ & \;\; j = 1 \\ 3(n-i)+4 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1,\dots,n \\ & \;\; j = 2 \\ 3(n-i)+3 & v\,; i = \left\lceil \frac{n}{2} \right\rceil + 1,\dots,n \\ & \;\; j = 3 \end{cases}$$

$$f(v_i v_{i,j}) = j \qquad ; i = 1 \quad j = 1,2,3$$

Since $w_f(v_i) = \sum_{v_j \epsilon N(v_i)} f(v_j) + \sum_{v_i v_{i,j} \epsilon I(v_i)} f(v_i v_{i,j})$, for $i = 1,2 \dots, n$, $j = 1,2,3$, $n \geq 5$ an odd positive integer, we obtain,

$w_f(v_{1,1}) = f(v_1) + f(v_1 v_{1,1}) = 1 + 1 = 2$
$w_f(v_{1,2}) = f(v_1) + f(v_1 v_{1,2}) = 1 + 2 = 3$
$w_f(v_{1,3}) = f(v_1) + f(v_1 v_{1,3}) = 1 + 3 = 4$
$w_f(v_{2,1}) = f(v_2) + f(v_2 v_{2,1}) = 4 + 1 = 5$

$$\vdots$$

$$\begin{aligned} w_f\left(v_{\left\lceil \frac{n}{2} \right\rceil + 1,1}\right) &= f\left(v_{\left\lceil \frac{n}{2} \right\rceil + 1}\right) + f\left(v_{\left\lceil \frac{n}{2} \right\rceil + 1} v_{\left\lceil \frac{n}{2} \right\rceil + 1,1}\right) \\ &= 5 + 3\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 3\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 5 \\ &= 3n + 1 \end{aligned}$$

$w_f(v_1) = f(v_n) + f(v_n v_1) + f(v_1 v_2) + f(v_2) + f(v_1 v_{1,1}) + f(v_1 v_{1,2}) + f(v_1 v_{1,3})$

$$
\begin{aligned}
&\quad +f(v_{1,1})+f(v_{1,2})+f(v_{1,3})\\
&= 20\\
w_f(v_2) &= f(v_1)+f(v_1v_2)+f(v_2v_3)+f(v_3)+f(v_2v_{2,1})+f(v_2v_{2,2})+f(v_2v_{2,3})\\
&\quad +f(v_{2,1})+f(v_{2,2})+f(v_{2,3})\\
&= 19\\
w(v_n) &= f(v_{n-1})+f(v_{n-1}v_n)+f(v_1v_n)+f(v_1)+f(v_nv_{n,1})+f(v_nv_{n,2})\\
&\quad +f(v_nv_{n,3})+f(v_{n,1})++f(v_{n,2})++f(v_{n,3})\\
&= 26\\
&\ \vdots
\end{aligned}
$$

$$
\begin{aligned}
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= f\left(v_{\lceil\frac{n}{2}\rceil+1,1}\right)+f\left(v_{\lceil\frac{n}{2}\rceil+1}v_{\lceil\frac{n}{2}\rceil+1,1}\right)+f\left(v_{\lceil\frac{n}{2}\rceil+1,2}\right)+f\left(v_{\lceil\frac{n}{2}\rceil+1}v_{\lceil\frac{n}{2}\rceil+1,2}\right)\\
&\quad +f\left(v_{\lceil\frac{n}{2}\rceil+1,3}\right)+f\left(v_{\lceil\frac{n}{2}\rceil+1}v_{\lceil\frac{n}{2}\rceil+1,3}\right)+f\left(v_{\lceil\frac{n}{2}\rceil}\right)+f\left(v_{\lceil\frac{n}{2}\rceil}v_{\lceil\frac{n}{2}\rceil+1}\right)+f\left(v_{\lceil\frac{n}{2}\rceil+2}\right)\\
&\quad +f\left(v_{\lceil\frac{n}{2}\rceil+1}v_{\lceil\frac{n}{2}\rceil+2}\right)\\
&= \frac{n+1}{2}-3+\left\lceil\frac{n}{10}\right\rceil+3\left(n-\left\lceil\frac{n}{2}\right\rceil+1\right)+5+\frac{n+1}{2}-3+\left\lceil\frac{n}{10}\right\rceil\\
&\quad +3\left(n-\left\lceil\frac{n}{2}\right\rceil+1\right)+4+\frac{n+1}{2}-3+\left\lceil\frac{n}{10}\right\rceil+3\left(n-\left\lceil\frac{n}{2}\right\rceil+1\right)+3+3\left\lceil\frac{n}{2}\right\rceil\\
&\quad -2+\frac{n+1}{2}-3+\left\lceil\frac{n}{10}\right\rceil+5+3\left(n-\left\lceil\frac{n}{2}\right\rceil+2\right)+\frac{n+1}{2}-3+\left\lceil\frac{n}{10}\right\rceil\\
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\left\lceil\frac{n}{10}\right\rceil+10n-17
\end{aligned}
$$

For $n = 1,3,5,7,9 \ mod\ 10$, respectively, we obtain

$$
\begin{aligned}
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\frac{n+9}{10}+10n-17=\frac{21n-25}{2}\\
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\frac{n+7}{10}+10n-17=\frac{21n-27}{2}\\
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\frac{n+5}{10}+10n-17=\frac{21n-29}{2}\\
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\frac{n+3}{10}+10n-17=\frac{21n-31}{2}\\
w_f\left(v_{\lceil\frac{n}{2}\rceil+1}\right) &= 5\frac{n+1}{10}+10n-17=\frac{21n-33}{2}
\end{aligned}
$$

Thus, for $i = 1,\dots,n$, we obtain the set of vertices weights is $w_f(v_i) = \{2,3,\dots,3n+1\}$. The vertex weights are distinct and formed a progressive arithmetic sequence, hence $k = \left\lceil\frac{3n+1}{2}\right\rceil$.

Theorem 6. Let C_m^n be a hairy cycle graph with $m = 4$, $n \geq 5$ an odd positive integer. If C_4^n has a distance vertex irregular total k-labeling then the total distance vertex irregularity strength denoted by $tdis(C_4^n)$ is

$$tdis(C_4^n) = \left\lceil\frac{4n+1}{2}\right\rceil$$

Proof. Let C_4^n be a hairy cycle graph, $V(C_4^n) = \{v_1, \dots, v_n, v_{1,1}, \dots, v_{1,4}, \dots, v_{n,1}, \dots, v_{n,4}\}$ and $E(C_4^n) = \{v_1v_2, \dots, v_nv_1, v_1v_{1,1,}, \dots, v_1v_{1,4}, \dots, v_nv_{n,1}, \dots, v_nv_{n,4}\}$ be the set of vertices and edges of C_4^n with $|V(C_4^n)| = 5n$ and $|E(C_4^n)| = 5n$. Graph C_4^n has minimum degree $\delta = 1$, and maximum degree $\Delta = 6$. Based on lemma 2, the lower bound of $tdis(C_4^n)$ is

$$tdis(C_4^n) \geq \left\lceil \frac{mn+1}{2} \right\rceil = \left\lceil \frac{4n+1}{2} \right\rceil$$

Define f, a distance vertex irregular total k-labeling on C_4^n as follows :

$$f(v_i) = \begin{cases} 4i-3 & ; i = 1,2, \dots, \left\lceil \frac{n}{2} \right\rceil \\ 7+4(n-i) & ; i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n \end{cases}$$

$$f(v_{i,j}) = \left\lceil \frac{2n}{3} \right\rceil - 3 \qquad ; i = 1, \dots, n \quad j = 1,2,3$$

$$f(v_iv_{i,j}) = \begin{cases} 4i-7 & ; i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil \\ & \ \ j = 1 \\ 4i-6 & ; i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil \\ & \ \ j = 2 \\ 4i-5 & ; i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil \\ & \ \ j = 3 \\ 4i-4 & ; i = 2, \dots, \left\lceil \frac{n}{2} \right\rceil \\ & \ \ j = 4 \end{cases}$$

$$f(v_iv_{i,j}) = \begin{cases} 4(n-i)+6 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n \\ & \ \ j = 1 \\ 4(n-i)+5 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n \\ & \ \ j = 2 \\ 4(n-i)+4 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n \\ & \ \ j = 3 \\ 4(n-i)+3 & ; i = \left\lceil \frac{n}{2} \right\rceil + 1, \dots, n \\ & \ \ j = 4 \end{cases}$$

$$f(v_iv_{i,j}) = j \qquad ; i = 1 \quad j = 1,2,3,4$$

Since $w_f(v_i) = \sum_{v_j \epsilon N(v_i)} f(v_j) + \sum_{v_iv_{i,j} \epsilon I(v_i)} f(v_iv_{i,j})$, for $i = 1,2, \dots, n$, $j = 1,2,3,4$, $n \geq 5$ an odd positive integer, we obtain,

$w_f(v_{1,1}) = f(v_1) + f(v_1v_{1,1}) = 1 + 1 = 2$

$w_f(v_{1,2}) = f(v_1) + f(v_1v_{1,2}) = 1 + 2 = 3$

$w_f(v_{1,3}) = f(v_1) + f(v_1v_{1,3}) = 1 + 3 = 4$

$$w_f(v_{1,4}) = f(v_1) + f(v_1 v_{1,4}) = 1 + 4 = 5$$
$$\vdots$$
$$\begin{aligned} w_f\left(v_{\lceil \frac{n}{2} \rceil+1,1}\right) &= f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+1,1}\right) \\ &= 7 + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 6 \\ &= 4n + 1 \end{aligned}$$

$$\begin{aligned} w_f(v_1) = {} & f(v_n) + f(v_n v_1) + f(v_1 v_2) + f(v_2) + f(v_1 v_{1,1}) + f(v_1 v_{1,2}) + f(v_1 v_{1,3}) \\ & + f(v_1 v_{1,4}) + f(v_{1,1}) + + f(v_{1,2}) + f(v_{1,3}) + + f(v_{1,4}) \\ = {} & 28 \end{aligned}$$

$$\begin{aligned} w_f(v_2) = {} & f(v_1) + f(v_1 v_2) + f(v_2 v_3) + f(v_3) + f(v_2 v_{2,1}) + f(v_2 v_{2,2}) + f(v_2 v_{2,3}) \\ & + f(v_2 v_{2,4}) + f(v_{2,1}) + f(v_{2,2}) + f(v_{2,3}) + f(v_{2,4}) \\ = {} & 26 \end{aligned}$$

$$\begin{aligned} w(v_n) = {} & f(v_{n-1}) + f(v_{n-1} v_n) + f(v_1 v_n) + f(v_1) + f(v_n v_{n,1}) + f(v_n v_{n,2}) \\ & + f(v_n v_{n,3}) + f(v_n v_{n,4}) + f(v_{n,1}) + f(v_{n,2}) + f(v_{n,3}) + + f(v_{n,4}) \\ = {} & 36 \\ & \vdots \end{aligned}$$

$$\begin{aligned} w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = {} & f\left(v_{\lceil \frac{n}{2} \rceil+1,1}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+1,1}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1,2}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+1,2}\right) \\ & + f\left(v_{\lceil \frac{n}{2} \rceil+1,3}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+1,3}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1,4}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+1,4}\right) \\ & + f\left(v_{\lceil \frac{n}{2} \rceil}\right) + f\left(v_{\lceil \frac{n}{2} \rceil} v_{\lceil \frac{n}{2} \rceil+1}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+2}\right) + f\left(v_{\lceil \frac{n}{2} \rceil+1} v_{\lceil \frac{n}{2} \rceil+2}\right) \\ = {} & \left\lceil \frac{2n}{3} \right\rceil - 3 + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 6 - 3 + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 5 - 3 \\ & + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 4 - 3 + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) + 3 + 4\left\lceil \frac{n}{2} \right\rceil - 3 - 3 - 7 \\ & + 4\left(n - \left\lceil \frac{n}{2} \right\rceil + 2\right) + \left\lceil \frac{2n}{3} \right\rceil - 3 \end{aligned}$$

$$w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = 12n - 28 + 6\left\lceil \frac{2n}{3} \right\rceil$$

Case 1. For $2n = 2, 14\ mod\ 24$

If $2n = 2\ mod\ 24$, then $w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = 12n - 28 + 6\left\lceil \frac{2n}{3} \right\rceil = 12n - 28 + 6\left(\frac{2n+1}{3}\right) = 12n - 22$ and if $2n = 14\ mod\ 24$, then $w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = 12n - 28 + 6\left\lceil \frac{2n}{3} \right\rceil = 12n - 28 + 6\left(\frac{2n+1}{3}\right) = 12n + 2$.

Case 2. For $2n = 6, 18\ mod\ 24$

If $2n = 6\ mod\ 24$, then we obtain $w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = 12n - 28 + 6\left\lceil \frac{2n}{3} \right\rceil = 12n - 28 + 6\left(\frac{2n}{3}\right) = 12n - 16$ and for $2n = 18\ mod\ 24$, we obtain $w_f\left(v_{\lceil \frac{n}{2} \rceil+1}\right) = 12n - 28 + 6\left\lceil \frac{2n}{3} \right\rceil = 12n - 28 + 6\left(\frac{2n}{3}\right) = 12n + 8$.

Case 3. For $2n = 10, 22 \; mod \; 24$

If $2n = 10 \; mod \; 24$, then $w_f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) = 12n - 28 + 6\left\lceil\frac{2n}{3}\right\rceil = 12n - 28 + 6\left(\frac{2n+2}{3}\right) = 12n - 4$ and for $2n = 22 \; mod \; 24$, we obtain $w_f\left(v_{\left\lceil\frac{n}{2}\right\rceil+1}\right) = 12n - 28 + 6 = 12n - 28 + 6\left(\frac{2n+2}{3}\right) = 12n + 20$.

Thus, for $i = 1, \ldots, n$, we obtain the set of vertices weight is $w_f(v_i) = \{2,3, \ldots ,4n + 1\}$. Every vertex weight is distinct and formed an arithmetic progressive, hence $k = \left\lceil\frac{4n+1}{2}\right\rceil$.

Results and Discussion

In the previous, we determine the lower bound of the total distance vertex irregularity strength and established the exact value of the total distance vertex irregularity strength for hairy cycle graph. The example of distance vertex irregular total k-labeling on C_m^n, $m = 2, n = 13$, described in Figure 2.

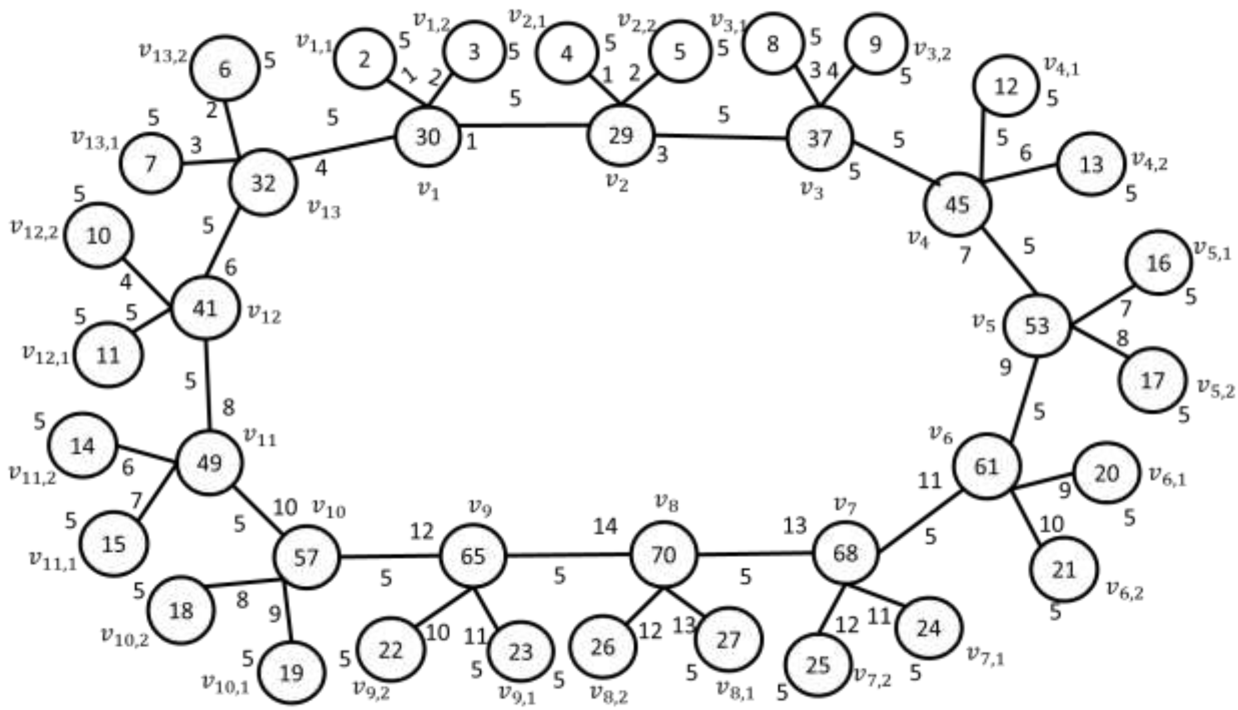

Figure 2. A Distance vertex irregular total k-labeling on C_m^n, $m = 2, n = 13$

To apply this labeling result in cryptography, we do the protocol of modified stream cipher with distance vertex irregular total k-labeling as follows,

1. Let P, be a set of plaintext (in ascii 127 code), $|P| = p$.
2. Choose $w(v_i)$; $i = 1, \ldots, n$ of distance vertex irregular total k-labeling on C_m^n
3. Transform the sequence to integer in modulo 127 then convert it to a 7 bit binary number
4. Use the keystream to encrypt the plaintext. If $p > n$ than then the keystream is reused from the beginning.

In Example 1, we perform the process of encryption in the modified stream cipher.

Example 1.
Using hairy cycle C_2^{13}, we choose the weight of vertices of C_2^{13} i.e. $w(v_1), w(v_2), \dots, w(v_n)$ and obtained a sequence of random number, $30, 29, 37, 45, 53, 61, 68, 70, 65, 57, 49, 41, 32$. Transforming the sequence integer to modulo 127 results in 0011110 0011101 0100101 0101101 0110101 0111101 1000100 1000110 1000001 0111001 0110001 0101001 0100000. Suppose the plaintext is 0101111 1110101 0111101 1010100 1001110 1000001 0110001 0110011, the encryption process is

0101111 1110101 0111101 1010100 1001110 1000001 0110001 0110011
0011110 0011101 0100101 0101101 0110101 0111101 1000100 1000110
____ $\oplus$
0110001 1101000 0011000 1111001 1111011 1111100 1110101 1110101

The sequence number is the ciphertext, obtained from the encryption process. The decryption process is
0110001 1101000 0011000 1111001 1111011 1111100 1110101 1110101
0011110 0011101 0100101 0101101 0110101 0111101 1000100 1000110
____ $\oplus$
0101111 1110101 0111101 1010100 1001110 1000001 0110001 0110011

This process reverses the ciphertext to plaintext.

Conclusions

The hairy cycle graph, C_m^n, has a distance vertex irregular total k-labeling. For $m = 2,3,4$, $n \geq 5$ an odd positive integer, the lower bound of total distance vertex irregularity strength of C_m^n is $tdis(C_m^n) \geq \left\lceil \frac{mn+1}{2} \right\rceil$. The values of total distance vertex irregularity strength $tdis(C_m^n)$ for $m = 2, 3, 4$ are $tdis(C_2^n) = \left\lceil \frac{2n+1}{2} \right\rceil$, $tdis(C_3^n) = \left\lceil \frac{3n+1}{2} \right\rceil$, and $tdis(C_4^n) = \left\lceil \frac{4n+1}{2} \right\rceil$, respectively. The formula of selected vertex weight, could be applied on cryptography as a keystream generator for stream cipher cryptography.

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